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The Laguerre–Gaussian series representation of two-dimensional fractional Fourier transform

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Abstract. In this paper the Laguerre–Gaussian (LG) series representation of the two-dimensional fractional Fourier transform is derived from conventional ordinary Fourier transform in polar coordinates. The kernel of this series representation is constituted by Laguerre–Gaussian functions, from which the series representation of a fractional Hankel transform can be easily derived. The connection between the gradient-index medium and the LG series representation is illustrated as an example of its applications.

The fractional Fourier transform (FrFT) was derived by Namias [1] as a new mathematical tool in order to deal with some problems in quantum mechanics. The key step in his derivation is to write the eigenvalue as the form of a semigroup operator after the introduction of fractional order in the eigenvalue equation of one-dimensional (1D) Fourier transform (FT). As a result, the integral representation of FrFT was heuristically defined as well as the series representation. The scope of its application was greatly widened after the notation of FrFT was introduced into optics by Ozaktas and Mendlovic [2–4] through a different approach, the *Gedankenexperiment* with gradient-index (GRIN) media. In other words, the expression, representing the propagation of light waves in the GRIN media, was verified to satisfy the properties of a semigroup. Therefore, this expression was considered as a two-dimensional (2D) FrFT whose kernel in rectangular coordinates is symbolized by a summation of Hermite–Gaussian (HG) functions and can be decomposed mathematically as a product of two 1D FrFT kernels. This result is regarded as the extension of 1D FrFT to the 2D case [5, 6], for the reason that HG functions are fundamentally the eigenfunctions for 1D FT. Conversely, the kernel of 2D FT in polar coordinates is of a 2D origin. The motivation for this paper is to derive a different, yet equivalent, series representation of the 2D FrFT form 2D FT in the plane-polar coordinates.

Let us begin with a two-dimensional Fourier transform in rectangular coordinates:

$$f_2(u_1, u_2) = \frac{1}{2\pi} \int_0^{+\infty} \mathrm{d}x_1 \int_0^{+\infty} \mathrm{d}x_2 \, f_1(x_1, x_2) \, \exp[\mathrm{i}(x_1 u_1 + x_2 u_2)]. \qquad (1)$$

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When $x_1 \equiv r_1 \cos\theta$, $x_2 \equiv r_1 \sin\theta$, $u_1 \equiv r_2 \cos\varphi$, and $u_2 \equiv r_2 \sin\varphi$, we express the FT in plane-polar coordinates:

$$f_2(r_2, \varphi) = \frac{1}{2\pi} \int_0^{+\infty} \mathrm{d}r_1\, r_1 \int_0^{2\pi} \mathrm{d}\theta\ \exp[\mathrm{i}\, r_1 r_2 \cos(\theta - \varphi)]\, f_1(r_1, \theta). \tag{2}$$

The FT and inverse FT can be written in the operator form:

$$f_2(r_2, \varphi) = F^1[f_1(r_1, \theta)]$$
$$f_1(r_1, \theta) = F^{-1}[f_2(r_2, \varphi)].$$

The operators F^1 and F^{-1}, denoting the 2D FT and its inverse transform, are complex conjugates of each other and satisfy the relations $F^{-1}F^1 = F^1F^{-1} = 1$.

The eigenvalue equation for operator F^1 is given by

$$F^1[\Psi_{pm}(r_1)\exp(\mathrm{i}m\theta)] = (-1)^p(\mathrm{i})^m\Psi_{pm}(r_2)\exp(\mathrm{i}m\varphi) \tag{3}$$

with the Laguerre–Gaussian (LG) function $\Psi_{pm}(r)$:

$$\Psi_{pm}(r) = \left[\frac{2(p!)}{(p+m)!}\right]^{1/2} \exp\left(-\tfrac{1}{2}r^2\right) r^m L_p^m(r^2) \tag{4}$$

where L_p^m is the generalized Laguerre polynomial. Three integral relations [9] are employed in the derivation of equations (3) and (4). Namely, the following two expressions are used to prove the eigenvalue equation:

$$\exp\left(-\tfrac{1}{2}x\right) x^{m/2}\, L_p^m(x) = \tfrac{1}{2}(-1)^p \int_0^{+\infty} \mathrm{d}y\ \exp\left(-\tfrac{1}{2}y\right) J_m(\sqrt{xy})\, y^{m/2}\, L_p^m(y).$$

In the above, let $y = r_1^2$, $x = r_2^2$, then it is proved that;

$$\Psi_{pm}(r_2) = (-1)^p \int_0^{+\infty} \mathrm{d}r_1\, r_1\, \Psi_{pm}(r_1)\, J_m(r_1 r_2).$$

Another integral relation to determine the normalized constant is

$$\int_0^{+\infty} \mathrm{d}x\ \exp(-x)\, x^m\, L_p^m(x)\, L_p^m(x) = \frac{(p+m)!}{p!}\delta_{pm}.$$

Then, rewrite equation (3) in a different form:

$$F^1[\Psi_{pm}(r_1)\exp(\mathrm{i}m\theta)] = \exp\left[\mathrm{i}(2p+m)\tfrac{1}{2}\pi \times 1\right]\Psi_{pm}(r_2)\exp(\mathrm{i}m\varphi). \tag{5}$$

We now assume that the FrFT operator F^a, satisfies the eigenvalue equation:

$$F^a[\Psi_{pm}(r_1)\exp(\mathrm{i}m\theta)] = \exp\left[\mathrm{i}(2p+m)\tfrac{1}{2}\pi \times a\right]\Psi_{pm}(r_2)\exp(\mathrm{i}m\varphi). \tag{6}$$

This equation implies that $\Psi_{pm}(r)\exp(\mathrm{i}m\theta)$ is the eigenfunction of the operator F^a with eigenvalue $\exp\left[\mathrm{i}(2p+m)\tfrac{1}{2}\pi \times a\right]$. As is well known, any square-integral function $g(r, \theta)$ can be expanded in terms of this eigenfunction:

$$g(r, \theta) = \sum_{m=0}^{\infty}\sum_{p=m}^{\infty} A_{pm}\Psi_{pm}(r)\exp(\mathrm{i}m\theta) \tag{7}$$

with

$$A_{pm} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \int_0^{+\infty} \mathrm{d}r\, r\, g(r, \theta)\, \Psi_{pm}(r)\ \exp(-\mathrm{i}m\theta). \tag{8}$$

In view of equations (6)–(8), we give the definition of the FrFT.

Definition. The LG series representation of a two-dimensional FrFT in polar coordinates is defined as

$$F^a[g(r,\theta)] = \sum_{m=0}^{\infty}\sum_{p=m}^{\infty} \exp\left[\mathrm{i}(2p+m)\tfrac{1}{2}\pi \times a\right] A_{pm}\Psi_{pm}(r)\exp(\mathrm{i}m\theta) \tag{9}$$

where $\Psi_{pm}(r)$ and A_{pm} satisfy equations (4) and (8), respectively.

The above relation is called the Laguerre–Gaussian series representation of a 2D FrFT, which is proved to be equivalent to the HG series representation defined by Ozaktas and Mendlovic. Nevertheless, in contrast to the HG series representation, the LG series representation is intrinsically two dimensional.

Consider a GRIN medium with the refractive-index distribution $n(r)$:

$$n^2(r) = n_1^2[1-(n_2/n_1)r^2]. \tag{10}$$

The scalar Helmholtz wave equation in cylindric coordinates [8] is written as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\boldsymbol{E}_{pm}(r,\theta,z) = 0 \tag{11}$$

where $k^2 = k_1^2 - k_2^2 r^2$ with $k = 2\pi n/\lambda$, $k_1 = 2\pi n_1/\lambda$ and $k_2 = 2\pi\sqrt{n_1 n_2}/\lambda$.

Denoting the propagating eigenmode, the solution $\boldsymbol{E}_{pm}(r,\theta,z)$ is the LG function defined in equations (12) and (13):

$$\boldsymbol{E}_{pm}(r,\theta,z) = \frac{2}{\omega^2}\Psi_{pm}\left(\frac{\sqrt{2}r}{\omega}\right)\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\beta_{pm}z) \tag{12}$$

with

$$\beta_{pm} = k_1\left[1 - \frac{1}{k_1}\sqrt{\frac{n_2}{n_1}}(4p+2m+2)\right]^{1/2} \simeq k_1 - \sqrt{\frac{n_2}{n_1}}(2p+m+1) \tag{13}$$

where $\omega = (2/k_1)^{1/2}(n_1/n_2)^{1/4}$.

The field $f_z(r,\theta)$ is the field $f_0(r,\theta)$ at plane $z=0$ after propagating a distance of z:

$$f_z(r,\theta) = \frac{2}{\omega^2}\sum_{m=0}^{\infty}\sum_{p=m}^{\infty} A'_{pm}\Psi_{pm}\left(\frac{\sqrt{2}r}{\omega}\right)\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\beta_{pm}z) \tag{14}$$

with

$$A'_{pm} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r\ r\ f_0(r,\theta)\frac{2}{\omega^2}\Psi_{pm}\left(\frac{\sqrt{2}r}{\omega}\right)\ \exp(-\mathrm{i}m\theta). \tag{15}$$

Defining $L = (\pi/2)\sqrt{n_1/n_2}$ and $a = z/L$, we have

$$f_z(r,\theta) = T[f_0(r,\theta)] = \frac{2}{\omega^2}\sum_{m=0}^{\infty}\sum_{p=m}^{\infty} A'_{pm}\Psi_{pm}\left(\frac{\sqrt{2}r}{\omega}\right)\exp(\mathrm{i}m\theta)\exp[-\mathrm{i}\beta_{pm}aL]. \tag{16}$$

According to equation (13),

$$\exp[-\mathrm{i}\beta_{pm}aL] = \exp\left[-\mathrm{i}k_1 aL + \mathrm{i}(2p+m+1)\tfrac{1}{2}\pi a\right]. \tag{17}$$

Changing the coordinates in equations (14)–(16) with $r' \equiv \sqrt{2}r/\omega$ and defining

$$g(r',\theta) \equiv g\left(\frac{\sqrt{2}r}{\omega},\theta\right) \equiv f_0(r,\theta) \tag{18}$$

thus,

$$T[f_0(r,\theta)] = \frac{2}{\omega^2}\sum_{m=0}^{\infty}\sum_{p=m}^{\infty} A'_{pm}\Psi_{pm}(r')\exp(\mathrm{i}m\theta)\exp[-\mathrm{i}\beta_{pm}aL] \tag{19}$$

with

$$A'_{pm} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r'\,r'\,g(r',\theta)\,\Psi_{pm}(r')\,\exp(-\mathrm{i}m\theta). \tag{20}$$

Finally, we obtain

$$T[f_0(r,\theta)] = \frac{2}{\omega^2}\exp\left(-\mathrm{i}k_1aL + \tfrac{1}{2}\mathrm{i}\pi a\right)F^a[g(r',\theta)]. \tag{21}$$

It is concluded from the above expression that the propagation of light waves in isotropic GRIN media can be described by the LG series representation of the 2D FrFT as well as the HG series representation. However, the HG series representation is a more general framework for the reason that it can also describe the elliptic GRIN media [10].

An input function $g(r,\theta)$ with rotational symmetry, has the form;

$$g(r,\theta) = g_0(r)\exp(\mathrm{i}m\theta). \tag{22}$$

Hence, a simplified form for F^a is

$$F^a[g_0(r)\exp(\mathrm{i}m\theta)] = \sum_{p=m}^{\infty}\exp\left[\mathrm{i}(2p+m)\tfrac{1}{2}\pi\times a\right]A_{pm}\Psi_{pm}(r)\exp(\mathrm{i}m\theta) \tag{23}$$

with

$$A_{pm} = \int_0^{+\infty}\mathrm{d}r\,r\,g_0(r)\,\Psi_{pm}(r).$$

Obviously there is only one summation in the expression that offers facilities in analysis and calculations. Another application of the LG representation is that the series representation of a fractional Hankel transform (FrHT) can be easily derived from it. In [11], there exists a simple relation between this FrFT operator F^a and an ath-order FrHT operator H_m^a, where m represents the mth-order Bessel function, namely

$$H_m^a = \exp\left(-\mathrm{i}ma\tfrac{1}{2}\pi\right)F^a. \tag{24}$$

From the above relation and by the employment of equations (8) and (9), the series representation of the FrHT can be defined as

$$H_m^a[f(r)] = \sum_{p=m}^{\infty}\exp(\mathrm{i}2pa)\,B_{pm}\,\Psi_{pm}(r) \tag{25}$$

with

$$B_{pm} = \int_0^{+\infty}\mathrm{d}r\,r\,f(r)\,\Psi_{pm}(r). \tag{26}$$

In summary, we have derived the series representation of 2D FrFT in plane-polar coordinates, which has advantages in studies of the physical systems with circular symmetry such as the isotropic GRIN fibre, circular laser cavities [7] and quantum systems [1].

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